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2004 J. Phys. A: Math. Gen. 37 8365

(http://iopscience.iop.org/0305-4470/37/34/012)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **37** (2004) 8365–8381 PII: S0305-4470(04)76148-X

Variational Hamiltonian treatment of partially reduced Yukawa-like models

A Duviryak[1] **and J W Darewych**[2]

[1] Department for Metal and Alloy Theory, Institute for Condensed Matter Physics of NAS of Ukraine, Lviv, UA-79011, Ukraine
[2] Department of Physics and Astronomy, York University, Toronto, Ontario, M3J 1P3, Canada

E-mail: duviryak@ph.icmp.lviv.ua and darewych@yorku.ca

Received 13 February 2004
Published 11 August 2004
Online at stacks.iop.org/JPhysA/37/8365
doi:10.1088/0305-4470/37/34/012

Abstract

We consider a variational method for deriving relativistic two- (and many-) body wave equations for interacting matter fields from partially reduced quantum field theory. The classical Lagrangian of the theory is reformulated by partially eliminating the mediating field by means of the covariant Green function. The reformulated Lagrangian contains time-nonlocal interaction terms in which the mediating-field Green function appears directly, sandwiched between the particle currents. The transition to the Hamiltonian formalism is implemented within an approximation scheme, on account of the time nonlocality. We consider an approximation, which is first order in the coupling constant. The system is quantized canonically. The variational principle is used to derive relativistic two-particle integral wave equations with a kernel, which include the 4D Fourier transform of the covariant Green function. We apply this approach to the Yukawa model and consider generalizations based on nonstandard field-theoretical models.

PACS numbers: 12.20.Ds, 03.65.Pm, 36.10.Dr, 11.10.St

1. Introduction

The use of the variational method in quantum field theory (QFT) is a promising approach to the relativistic bound state problem. The input in this method is a Hamiltonian for the system and a trial state which should be chosen appropriately for the problem considered. The choice of trial state in the variational method is more like guess work than a formal procedure, especially in QFT, where channel wavefunctions stand for variational parameters [1, 2]. In principle, the more channels are included in the trial state, the better the approximation, but the more complicated the set of coupled wave equations that is obtained as a result.

For processes in which there are no physical bosons of the field mediating the interaction (or where their interaction with the system is negligible), the treatment of the system can be simplified considerably. The idea lies in the elimination of the mediating field at the classical level by means of covariant Green functions [3–6]. The reformulated theory includes only the field variables of the matter fields but not those of the mediating field. Consequently, the set of trial states at the quantum level, needed to describe the system, can be smaller compared with the original (non-reformulated) theory. Thus the combination of the variational method with the partial reduction of the QFT becomes an effective tool for describing bound and quasi-bound states [3, 5, 6].

However, the reformulation leads to another problem. The reformulated Lagrangian contains nonlocal interaction terms in which the mediating-field Green functions appear directly, sandwiched between the particle currents. The Hamiltonization of such Lagrangians is not a straightforward procedure. In the papers cited above the Hamiltonization procedure was performed by means of prescriptions, which are not rigorously substantiated, or have restricted application.

In the present paper we consider the problem of consistent Hamiltonization of nonlocal partially reduced Lagrangians. The approach is based on the general procedure of Hamiltonization developed in the literature for nonlocal Lagrangian mechanics [7, 8]. Specifically the procedure is based on expansions in the coupling constant. Special attention is paid to the Noether conserved quantities which correspond to the Poincaré invariance of the nonlocal Lagrangian.

In the present paper we first illustrate this procedure on the partially reduced scalar Yukawa model, for which we obtain a variational wave equation for two-particle states in first-order approximation. Thereafter we consider a generalization of the Yukawa model, in which the Green function of the Klein–Gordon equation (for the mediating field) is replaced by any symmetric Poincaré-invariant kernel appropriate for different types of interactions. The interaction kernel may be chosen phenomenologically, or derived from some effective QFT. It may arise, also, from nonstandard classical field theories, the quantization of which is difficult or inconsistent. We shall consider a few such examples in this paper. Two of them are based on higher derivative field theories that lead to confining interactions. Others originate from nonlocal field theory.

2. Yukawa model in standard and partially reduced formulations

We proceed from the classical action integral $I = \int \mathrm{d}^4x\mathcal{L}(x)$, where the Lagrangian density $\mathcal{L} = \sum_{a=1}^{2}\mathcal{L}_a + \mathcal{L}_\chi + \mathcal{L}_\mathrm{Y}$ consists of the following terms: $\mathcal{L}_a$ are standard free-field terms corresponding to complex scalar 'matter' fields $\phi_a(x)$ $(a = 1, 2)$ with rest masses m_a, $\mathcal{L}_\chi$ is the free-field term of a real scalar mediating field $\chi(x)$ and $\mathcal{L}_\mathrm{Y} = \rho\chi$ is the Yukawa interaction term, where

$$\rho = -\sum_{a=1}^{2} g_a\phi_a^*\phi_a. \tag{2.1}$$

For the standard Yukawa model $\mathcal{L}_\chi = \frac{1}{2}\{(\partial_\mu\chi)(\partial^\mu\chi) - \mu^2\chi^2\}$, where μ is the rest mass of the quanta of χ.

The variation of the action I leads to the coupled set of the Euler–Lagrange equations,

$$\big(\Box + m_a^2\big)\phi_a = -g_a\phi_a\chi, \qquad \big(\Box + m_a^2\big)\phi_a^* = -g_a\phi_a^*\chi, \tag{2.2}$$

$$(\Box + \mu^2)\chi = \rho. \tag{2.3}$$

Equation (2.3) can be solved exactly [4, 5]: $\chi(x) = \chi_0(x) + \int \mathrm{d}^4x' G(x, x'; \mu)\rho(x')$, where $\chi_0(x)$ is the solution of the homogeneous Klein–Gordon equation for the χ field, and $G(x, x', \mu)$ is the Green function of the Klein–Gordon equation. Using this solution in the equations (2.2) one obtains a coupled set of integro-differential equations for the fields $\phi_a(x)$ and $\phi_a^*(x)$, which we shall refer to as *partially-reduced field equations*.

Henceforth we shall put $\chi_0(x) = 0$, that is, we shall not consider processes involving physical 'chions' (quanta of the mediating field). In addition, we choose the Green function to be symmetric: $G(x, x'; \mu) = D_0(x - x'; \mu) = D_0(x' - x; \mu)$. This choice guarantees the existence of a variational principle for the partially-reduced field equations [9], which is very important for the quantization of the model. Therefore, the solution for $\chi(x)$ reads

$$\chi(x) = [D_0 * \rho](x) \equiv \int \mathrm{d}^4x' D_0(x - x'; \mu)\rho(x'). \tag{2.4}$$

Using this solution in the original action integral, we obtain (modulo surface terms) a modified action which leads to the partially-reduced field equations.

At this point we generalize the original Yukawa model and replace the Green function $D_0(x - x'; \mu)$ by an arbitrary symmetric Poincaré-invariant kernel $K(x - x') = K(x' - x)$. The modified Lagrangian density,

$$\begin{aligned}\mathcal{L} &= \sum_{a=1}^{2} \mathcal{L}_a + \mathcal{L}_{\mathrm{int}} \\ &= \sum_{a=1}^{2} \left\{(\partial_\mu \phi_a^*)(\partial^\mu \phi_a) - m_a^2 \phi_a^* \phi_a\right\} + \frac{1}{2}\int \mathrm{d}^4x' \rho(x) K(x - x')\rho(x'),\end{aligned} \tag{2.5}$$

can be useful for the description of effective interactions of quarks and hadrons. For example, if $K(x - x')$ is the solution of the equations $\Box^2 K(x - x') = \delta(x - x')$, the model leads to a confining interaction (see section 9).

3. Conserved quantities

The Poincaré-invariance of the Yukawa model leads to the existence of conserved energy–momentum tensor $T^{\mu\nu}$ and angular-momentum tensor $M^{\mu\lambda\sigma} = T^{\mu\lambda}x^\sigma - T^{\mu\sigma}x^\lambda$. They generate corresponding conserved quantities: the 4-momentum $P^\mu = \int \mathrm{d}^3x\, T^{0\mu}$ and the angular momentum $M^{\lambda\sigma} = \int \mathrm{d}^3x\, M^{0\lambda\sigma}$ (here $\mu, \nu, \lambda, \sigma = 0, \ldots, 3$). The energy–momentum tensor can be written in the form

$$T^{\mu\nu} = \sum_{a=1}^{2} T_a^{\mu\nu} + T_\chi^{\mu\nu} + T_{\mathrm{Y}}^{\mu\nu}, \tag{3.1}$$

where

$$T_a^{\mu\nu} = \{(\partial^\mu \phi_a^*)(\partial^\nu \phi_a) + (\partial^\nu \phi_a^*)(\partial^\mu \phi_a)\} - \eta^{\mu\nu}\mathcal{L}_a, \tag{3.2}$$

$$T_\chi^{\mu\nu} = (\partial^\mu \chi)(\partial^\nu \chi) - \eta^{\mu\nu}\mathcal{L}_\chi, \tag{3.3}$$

$$T_{\mathrm{Y}}^{\mu\nu} = -\eta^{\mu\nu}\rho\chi, \tag{3.4}$$

and $\|\eta_{\mu\nu}\| = \mathrm{diag}(+, -, -, -)$ is the Minkowski metric.

Substitution of the solution (2.4) into equations (3.1)–(3.4) yields the conserved quantities for the partially reduced model. These expressions are not satisfactory for a number

of reasons. First, the energy–momentum tensor (3.1) includes terms which are quadratic in χ. These terms are rather cumbersome, and it is not evident that products of distributions $D_0(x-x')$ in these terms are well defined. Second, expressions (3.1)–(3.4) are not valid for the generalized model (2.5). Therefore, we discuss next the procedure for obtaining the appropriate expressions for conserved quantities.

By making use of the field equation (2.3) we obtain the equality $\partial_\mu T^{\mu\nu}_\chi = \rho\partial^\nu\chi$, which, upon integration over all 3-space, yields,

$$\int \mathrm{d}^3x\, \partial_\mu T^{\mu\nu}_\chi = \frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}^3x\, T^{0\nu}_\chi = \int \mathrm{d}^3x\, \rho\partial^\nu\chi. \tag{3.5}$$

Integrating further over time and using equation (2.4) leads to the result

$$\begin{aligned}\int \mathrm{d}^3x\, T^{0\nu}_\chi &= \int_{-\infty}^{t}\mathrm{d}x^0\int \mathrm{d}^3x\,\rho\partial^\nu\chi \\ &= \frac{1}{2}\left[\int_{-\infty}^{t}\int_{t}^{\infty} - \int_{t}^{\infty}\int_{-\infty}^{t}\right]\mathrm{d}x^0\,\mathrm{d}x'^0\int \mathrm{d}^3x\int \mathrm{d}^3x'\rho(x)\{\partial^\nu D_0(x-x')\}\rho(x').\end{aligned} \tag{3.6}$$

The last equality holds by virtue of the skew-symmetry of the kernel: $\partial^\nu D_0(x-x') = -\partial'^\nu D_0(x'-x)$ which, in turn, is a consequence of the symmetry of $D_0(x-x')$.

The calculation of other contributions to the 4-momentum is straightforward and, in particular,

$$\int \mathrm{d}^3x\, T^{0\mu}_{\mathrm{Y}} = -\eta^{0\mu}\int \mathrm{d}^3x\int \mathrm{d}^4x'\rho(x)D_0(x-x')\rho(x'). \tag{3.7}$$

The 4-momentum is now linear in the chion propagator. Moreover, by replacing $D_0(x-x')$ with $K(x-x')$ in equations (3.6) and (3.7) we obtain the conserved 4-momentum for the generalized model defined by the Lagrangian density (2.5).

The treatment of the angular momentum is quite similar but more cumbersome. We omit the details and write down the final expressions for 4-momentum and angular momentum of the generalized model:

$$\begin{aligned}P^\mu(t) = \sum_{a=1}^{2}\int \mathrm{d}^3x\, T^{0\mu}_a(x)|_{x^0=t} - \eta^{0\mu}\int \mathrm{d}^3x\int \mathrm{d}^4x'\rho(x)K(x-x')\rho(x')|_{x^0=t} \\ - \int \mathrm{d}^4x\int \mathrm{d}^4x'\,\Xi(x^0-t,x'^0-t)\rho(x)\{\partial^\nu K(x-x')\}\rho(x'),\end{aligned} \tag{3.8}$$

$$\begin{aligned}M^{\lambda\sigma}(t) = \sum_{a=1}^{2}\int \mathrm{d}^3x\, T^{0[\lambda}_a(x)x^{\sigma]}|_{x^0=t} - \int \mathrm{d}^3x\int \mathrm{d}^4x'\rho(x)\eta^{0[\lambda}x^{\sigma]}K(x-x')\rho(x')|_{x^0=t} \\ - \int \mathrm{d}^4x\int \mathrm{d}^4x'\,\Xi(x^0-t,x'^0-t)\rho(x)\{\partial^{[\lambda}K(x-x')x^{\sigma]}\}\rho(x').\end{aligned} \tag{3.9}$$

We have used the notation $a^{[\mu}b^{\nu]} \equiv a^\mu b^\nu - a^\nu b^\mu$, and

$$\Xi(t,s) \equiv \theta(t)\theta(-s) - \theta(-t)\theta(s) = \tfrac{1}{2}(\operatorname{sign} t - \operatorname{sign} s), \tag{3.10}$$

in equations (3.9) and (3.10), where $\theta(t)$ is the Heaviside step function. We note that, in deriving equation (3.9), we exploited the skew-symmetry of the kernel $\partial^{[\lambda}K(x-x')x^{\sigma]} = -\partial'^{[\lambda}K(x'-x)x'^{\sigma]}$, which follows from the symmetry and the Poincaré-invariance of $K(x-x')$.

4. Hamiltonian structure of the nonlocal Lagrangian

The interaction term $\mathcal{L}_{\text{int}}$ in the Lagrangian density (2.5) is not a function but a functional of field variables. Thus it represents a nonlocal field-theoretical model.

Nonlocal field theories arose in attempts to remove ultraviolet divergences [10, 11] and their study has developed into an established branch of theoretical physics [12]. At present it encompasses a variety of models describing mostly the strong interaction of particles [13, 14], renormalization methods [15, 16], etc. Results are formulated mostly in the functional integration language and the treatment is rather complicated.

In this paper we use a treatment which is maximally close to the canonical quantization, i.e., we put the model into Hamiltonian form at the classical level and then perform a quantization. This permits us to separate peculiarities and some difficulties of the model due to its nonlocal nature from other general quantum features.

Because of the nonlocality, the transition to the Hamiltonian formalism cannot be performed in the standard way. Thus we will use the procedure which was developed for Hamiltonization in the formalism of Fokker action integrals [7] and then generalized to the case of arbitrary nonlocal Lagrangians [8].

The idea of Hamiltonization of nonlocal Lagrangians is the following. Suppose the Lagrangian $L(t,[q])$ is a functional of the dynamical variable $q(t)$. One replaces this variable by a 'field' variable $Q(t,\lambda)$ defined on the two-dimensional (time) $\times$ ('position') space, while the new Lagrangian $L(t,Q)$ is considered to be local in the time variable t and nonlocal in the 'position' variable λ. Simultaneously, the constraint $(\partial_t - \partial_\lambda)Q(t,\lambda) = 0$ is imposed. The transition to the Hamiltonian description of this new system is performed according to Dirac's constraint theory [17]. From the constraint we have $Q(t,\lambda) = q(t+\lambda)$. Moreover, the Hamiltonian flow is tangential to the submanifold of the phase space defined by the original Euler–Lagrange equation $\delta\{\int \mathrm{d}t' L(t',[q])\}/\delta q(t) = 0$. Thus the symplectic structure and the Hamiltonian can be pulled back to this submanifold.

In the case of field theory, the original fields $\phi(x) = \phi(x^0,\boldsymbol{x})$ on the Minkowski (1+3)-space are replaced by auxiliary variables on the (2+3)-space, which together with the Hamiltonian structure, are then reduced back onto the Minkowski space. Applying the above procedure with obvious modifications to the present case of field theory we write down expressions for the Liouville form and the Hamiltonian:

$$\Theta(t) = \int \mathrm{d}^4x \int \mathrm{d}^4x' \,\Xi(x^0 - t, x'^0 - t) \times \sum_{a=1}^{2} \{\mathcal{E}_a(x',x;[\phi])\tilde{\delta}\phi_a(x) + \mathcal{E}_{*a}(x',x;[\phi])\tilde{\delta}\phi_a^*(x)\}, \tag{4.1}$$

$$H(t) = \int \mathrm{d}^4x \int \mathrm{d}^4x' \,\Xi(x^0 - t, x'^0 - t) \times \sum_{a=1}^{2} \{\mathcal{E}_a(x',x;[\phi])\dot{\phi}_a(x) + \mathcal{E}_{*a}(x',x;[\phi])\dot{\phi}_a^*(x)\} - L(t); \tag{4.2}$$

here $\tilde{\delta}\phi_a(x)$ is the functional exterior differential of the field variables [8], $\dot{\phi}_a(x) = \partial\phi_a(x)/\partial x^0$,

$$\mathcal{E}_a(x,x';[\phi]) = \frac{\delta\mathcal{L}(x)}{\delta\phi_a(x')}, \qquad \mathcal{E}_{*a}(x,x';[\phi]) = \frac{\delta\mathcal{L}(x)}{\delta\phi_a^*(x')}, \tag{4.3}$$

and

$$L(t) = \int \mathrm{d}^3x\, \mathcal{L}(x)|_{x^0=t} \tag{4.4}$$

is the Lagrangian of the system.

The Liouville form determines the symplectic form, $\Omega = \tilde{\delta}\Theta$, and hence the Poisson brackets. The Hamiltonian H generates the evolution of the system in terms of the Poisson brackets. We note that the Hamiltonian (4.2), calculated for the Lagrangian density (2.5), coincides with the energy $E = P^0$ given in equation (3.8).

Equations (4.1) and (4.2) have only a formal meaning until we perform the time integrations explicitly. To do this we need the explicit solution of the partially reduced field equations (cf below equation (2.3)). Since these equations are rather complicated and not solvable exactly we shall resort to the approximation scheme described below.

5. Perturbation scheme and the first-order approximation

We shall use a formal series expansion in the coupling constant. Replacing the coupling constants g_a by $g_a\sqrt{\varepsilon}$ in the partially-reduced field equations (i.e., in the equations (2.2) with rhs expressed by equations (2.4) and (2.1)) we seek a solution in the form $\phi_a(x) = \sum_{n=0}^{\infty} \varepsilon^n \phi_a^{(n)}(x)$. Equating the coefficients of the powers of ε to zero we arrive at the infinite chain of coupled equations:

$$\left(\Box + m_a^2\right)\phi_a^{(n)}(x) = J_a^{(n)}(x), \qquad n = 0, 1, \ldots, \tag{5.1}$$

where $J_a^{(0)}(x) = 0$ while $J_a^{(n)}$ $(n \geqslant 1)$ are functionals of $\phi_a^{(m)}$ with $m \leqslant n-1$. Thus equations (5.1) form a hierarchy, which can be solved iteratively, from $n = 0$.

The equation for $\phi_a^{(0)}$ is the homogeneous Klein–Gordon equation, the solution of which is represented by plane waves. All higher-order functions $\phi_a^{(n)}$ can be found using the Green function method:

$$\phi_a^{(n)}(x) = \int \mathrm{d}^4x' G_a(x,x') J_a^{(n)}(x'), \tag{5.2}$$

where $G_a(x,x') = G(x,x';m_a)$ is the Green function of the Klein–Gordon equation. It is convenient to choose this function as follows:

$$G_a(x,x') = D_0(x-x';m_a) + \tfrac{1}{2}\,\mathrm{sign}\,x'^0 D(x-x';m_a), \tag{5.3}$$

where $D(x-x';m_a)$ is the Pauli–Jordan function [18]. Then the solutions $\phi_a^{(n)}$ for all $n \geqslant 1$ are tangent to $\phi_a^{(0)}$ at $x^0 = 0$:

$$\phi_a^{(n)}(0,\boldsymbol{x}) = 0, \qquad \dot{\phi}_a^{(n)}(0,\boldsymbol{x}) = 0, \quad n \geqslant 1, \tag{5.4}$$

so that

$$\phi_a(\boldsymbol{x}) \equiv \phi_a(0,\boldsymbol{x}) = \phi_a^{(0)}(0,\boldsymbol{x}), \qquad \dot{\phi}_a(\boldsymbol{x}) \equiv \dot{\phi}_a(0,\boldsymbol{x}) = \dot{\phi}_a^{(0)}(0,\boldsymbol{x}). \tag{5.5}$$

Using the expansion series for $\phi_a(\boldsymbol{x})$ on the rhs of (4.1), (4.2) and (3.8) yields series for the Liouville form Θ, the Hamiltonian H and the momentum $\boldsymbol{P} = (P^i, i = 1, 2, 3)$:

$$\Theta = \sum_{n=0}^{\infty} \varepsilon^n \Theta^{(n)}, \qquad H = \sum_{n=0}^{\infty} \varepsilon^n H^{(n)}, \qquad \boldsymbol{P} = \sum_{n=0}^{\infty} \varepsilon^n \boldsymbol{P}^{(n)}. \tag{5.6}$$

One can calculate the symplectic form and other conserved quantities similarly. After all the coefficients of the formal series are calculated (to the desired power), the parameter ε should be replaced by 1.

Since the quantities in (5.6) are evaluated at the same instant of time, we put $t = 0$. This corresponds to the Schrödinger picture at the quantum level.

It is easy to show using (5.5) that in zero-order approximation we have

$$\Theta^{(0)} \equiv \Theta^{\text{free}}[\phi] = \sum_{a=1}^{2} \int \mathrm{d}^3x \, \{\dot\phi_a^*(\boldsymbol{x})\tilde\delta\phi_a(\boldsymbol{x}) + \dot\phi_a(\boldsymbol{x})\tilde\delta\phi_a^*(\boldsymbol{x})\}, \tag{5.7}$$

$$H^{(0)} \equiv H^{\text{free}}[\phi] = \sum_{a=1}^{2} \int \mathrm{d}^3x \int \left\{\dot\phi_a^*(\boldsymbol{x})\dot\phi_a(\boldsymbol{x}) + (\nabla\phi_a^*(\boldsymbol{x}))\cdot(\nabla\phi_a(\boldsymbol{x})) + m_a^2\phi_a^*(\boldsymbol{x})\phi_a(\boldsymbol{x})\right\}, \tag{5.8}$$

$$\boldsymbol{P}^{(0)} \equiv \boldsymbol{P}^{\text{free}}[\phi] = -\sum_{a=1}^{2} \int \mathrm{d}^3x \, \{\dot\phi_a^*(\boldsymbol{x})\nabla\phi_a(\boldsymbol{x}) + \dot\phi_a(\boldsymbol{x})\nabla\phi_a^*(\boldsymbol{x})\}, \tag{5.9}$$

We note that the Liouville form and thus the symplectic form have the standard canonical expressions in this approximation. Thus the variables $(\phi_a(\boldsymbol{x}), \dot\phi_a(\boldsymbol{x}))$ and $(\phi_a^*(\boldsymbol{x}), \dot\phi_a^*(\boldsymbol{x}))$ form canonically conjugate pairs, which parametrize the phase space.

The first-order correction terms can be written as follows:

$$\Theta^{(1)} \equiv \Theta^{\text{nc}}[\phi] = \frac{1}{2}\int \mathrm{d}^4x \int \mathrm{d}^4x' \, \Xi(x^0, x'^0)\rho^{(0)}(x')K(x'-x)\tilde\delta\rho^{(0)}(x), \tag{5.10}$$

$$\begin{aligned} H^{(1)} \equiv H^{\text{nc}}[\phi] + H^{\text{int}}[\phi] &= \frac{1}{2}\int \mathrm{d}^4x \int \mathrm{d}^4x' \, \Xi(x^0, x'^0)\rho^{(0)}(x')K(x'-x)\dot\rho^{(0)}(x) \\ &\quad - \frac{1}{2}\int \mathrm{d}^3x \int \mathrm{d}^4x' \rho^{(0)}(x')K(x'-x)\rho^{(0)}(x)|_{x^0=0}, \end{aligned} \tag{5.11}$$

$$\boldsymbol{P}^{(1)} \equiv \boldsymbol{P}^{\text{nc}}[\phi] = -\frac{1}{2}\int \mathrm{d}^4x \int \mathrm{d}^4x' \, \Xi(x^0, x'^0)\rho^{(0)}(x')K(x'-x)\nabla\rho^{(0)}(x). \tag{5.12}$$

In this approximation the variables $\phi_a, \dot\phi_a, \phi_a^*, \dot\phi_a^*$ are no longer canonical due to the term $\Theta^{(1)} \equiv \Theta^{\text{nc}}$ of the Liouville form (this is indicated by the superscript 'nc'). The terms H^{nc} and $\boldsymbol{P}^{\text{nc}}$ have a similar nature, as is seen from their analogous structure.

In order to calculate $\Theta^{(1)}$ and $H^{(1)}$ explicitly it is convenient to transform each complex field into a pair of real fields:

$$\phi_a = \tfrac{1}{\sqrt{2}}(\phi_{a1} + \mathrm{i}\phi_{a2}), \qquad \phi_a^* = \tfrac{1}{\sqrt{2}}(\phi_{a1} - \mathrm{i}\phi_{a2}). \tag{5.13}$$

In zero-order approximation we obtain

$$\phi_{a\alpha}^{(0)}(x) = \frac{1}{(2\pi)^{3/2}} \sum_{A=\pm} \int \frac{\mathrm{d}^3k}{\sqrt{k_{a0}}} a_{a\alpha}^A(\boldsymbol{k})\, \mathrm{e}^{\mathrm{i}Ak_a\cdot x}, \qquad a = 1, 2, \ \alpha = 1, 2, \tag{5.14}$$

where $k_{a0} = \sqrt{m_a^2 + \boldsymbol{k}^2}$ and $k_a = (k_{a0}, \boldsymbol{k})$.

For brevity we replace the double subscript $a\alpha$ by a single subscript a ($a = 1, \ldots, 4$) in intermediate calculations. In this notation,

$$\begin{aligned} \Theta^{\text{free}} &= \sum_a \int \mathrm{d}^3x \, \dot\phi_a(\boldsymbol{x})\tilde\delta\phi_a(\boldsymbol{x}) \\ &= \frac{\mathrm{i}}{2}\sum_a \sum_{AB} \int \mathrm{d}^3k \, A a_a^A(\boldsymbol{k})\tilde\delta a_a^B(-AB\boldsymbol{k}), \end{aligned} \tag{5.15}$$

$$H^{\text{free}} = \frac{1}{2}\sum_a \int \mathrm{d}^3x \left\{\dot\phi_a^2(\boldsymbol{x}) + (\nabla\phi_a(\boldsymbol{x}))^2 + m_a^2\phi_a^2(\boldsymbol{x})\right\}$$
$$= \frac{1}{2}\sum_a \sum_A \int \mathrm{d}^3k\, k_{a0} a_a^A(\boldsymbol{k}) a_a^{-A}(\boldsymbol{k}), \tag{5.16}$$

$$\boldsymbol{P}^{\text{free}} = -\sum_a \int \mathrm{d}^3x\, \dot\phi_a(\boldsymbol{x})\nabla\phi_a(\boldsymbol{x})$$
$$= \frac{1}{2}\sum_a \sum_A \int \mathrm{d}^3k\, \boldsymbol{k} a_a^A(\boldsymbol{k}) a_a^{-A}(k); \tag{5.17}$$

hereafter roman capital indices $A, B, C, \ldots$ run over $+, -$. In the new notation the function $\rho(x)$, equation (2.3), takes on the form

$$\rho(x) = -\frac{1}{2}\sum_a g_a \phi_a^2(x), \tag{5.18}$$

and equations (5.10)–(5.12) remain valid.

Let us consider the first-order corrections to Θ and H. In momentum representation they take the form

$$\Theta^{\text{nc}} = -\frac{\mathrm{i}}{2}\sum_{ab}\sum_{ABCD}\int \mathrm{d}^3k\,\mathrm{d}^3q\,\mathrm{d}^3u\,\mathrm{d}^3v\, S_{ab}^{ABCD}(\boldsymbol{k},\boldsymbol{q},\boldsymbol{u},\boldsymbol{v})\big[\tilde\delta a_a^A(\boldsymbol{k})\big] a_a^B(\boldsymbol{q}) a_b^C(\boldsymbol{u}) a_b^D(\boldsymbol{v}), \tag{5.19}$$

$$H^{\text{nc}} = \frac{1}{2}\sum_{ab}\sum_{ABCD}\int \mathrm{d}^3k\,\mathrm{d}^3q\,\mathrm{d}^3u\,\mathrm{d}^3v\, S_{ab}^{ABCD}(\boldsymbol{k},\boldsymbol{q},\boldsymbol{u},\boldsymbol{v}) A k_{a0} a_a^A(\boldsymbol{k}) a_a^B(\boldsymbol{q}) a_b^C(\boldsymbol{u}) a_b^D(\boldsymbol{v}), \tag{5.20}$$

$$H^{\text{int}} = \frac{1}{2}\sum_{ab}\sum_{ABCD}\int \mathrm{d}^3k\,\mathrm{d}^3q\,\mathrm{d}^3u\,\mathrm{d}^3v\, T_{ab}^{ABCD}(\boldsymbol{k},\boldsymbol{q},\boldsymbol{u},\boldsymbol{v}) a_a^A(\boldsymbol{k}) a_a^B(\boldsymbol{q}) a_b^C(\boldsymbol{u}) a_b^D(\boldsymbol{v}). \tag{5.21}$$

Here

$$S_{ab}^{ABCD}(\boldsymbol{k},\boldsymbol{q},\boldsymbol{u},\boldsymbol{v}) = \frac{g_a g_b}{16(2\pi)^3}\frac{\delta^{(3)}(A\boldsymbol{k}+B\boldsymbol{q}+C\boldsymbol{u}+D\boldsymbol{v})}{\sqrt{k_{a0}q_{a0}u_{b0}v_{b0}}}$$
$$\times \mathcal{P}\frac{\tilde K(Ak_a + Bq_a) - \tilde K(Cu_b + Dv_b)}{Ak_{a0} + Bq_{a0} + Cu_{b0} + Dv_{b0}}, \tag{5.22}$$

$$T_{ab}^{ABCD}(\boldsymbol{k},\boldsymbol{q},\boldsymbol{u},\boldsymbol{v}) = -\frac{g_a g_b}{16(2\pi)^3}\frac{\delta^{(3)}(A\boldsymbol{k}+B\boldsymbol{q}+C\boldsymbol{u}+D\boldsymbol{v})}{\sqrt{k_{a0}q_{a0}u_{b0}v_{b0}}}\tilde K(Ak_a + Bq_a), \tag{5.23}$$

$\mathcal{P}$ stands for principal value, and

$$\tilde K(k) = \int \mathrm{d}^4x\, \mathrm{e}^{-\mathrm{i}k\cdot x} K(x). \tag{5.24}$$

The correction $\boldsymbol{P}^{\text{nc}}$ to the momentum can be easily obtained by means of the replacement $\tilde\delta a_a^A(\boldsymbol{k}) \to \mathrm{i}A\boldsymbol{k} a_a^A(\boldsymbol{k})$ in equation (5.19) for Θ^{nc}.

As pointed out above, the field variables in both the position and momentum representations are not canonical because of the first-order term of the Liouville form Θ^{nc}. Let us perform a transformation of variables $a \to \underline{a}$ such that the new variables $\underline{a}$ become canonical with respect to this form, at least to the accuracy required:

$$\Theta^{\text{free}}[a] + \varepsilon\Theta^{\text{nc}}[a] = \Theta^{\text{free}}[\underline{a}] + \mathrm{O}(\varepsilon^2). \tag{5.25}$$

Use of the ansatz

$$a_a^A(\boldsymbol{k}) = \underline{a}_a^A(\boldsymbol{k}) + \varepsilon \sum_b \sum_{BCD} \int \mathrm{d}^3q\, \mathrm{d}^3u\, \mathrm{d}^3v\, R_{ab}^{ABCD}(\boldsymbol{k}, \boldsymbol{q}, \boldsymbol{u}, \boldsymbol{v}) \underline{a}_a^B(\boldsymbol{q}) \underline{a}_b^C(\boldsymbol{u}) \underline{a}_b^D(\boldsymbol{v}) + \mathrm{O}(\varepsilon^2). \tag{5.26}$$

in (5.25) leads to the equation

$$\sum_F (F - A) R_{ab}^{FBCD}(-AF\boldsymbol{k}, \boldsymbol{q}, \boldsymbol{u}, \boldsymbol{v}) = S_{ab}^{ABCD}(\boldsymbol{k}, \boldsymbol{q}, \boldsymbol{u}, \boldsymbol{v}) \tag{5.27}$$

for the adjustable function $R_{ab}^{FBCD}(\boldsymbol{k}, \boldsymbol{q}, \boldsymbol{u}, \boldsymbol{v})$, which has the solution

$$R_{ab}^{ABCD}(\boldsymbol{k}, \boldsymbol{q}, \boldsymbol{u}, \boldsymbol{v}) = \frac{A}{2} S_{ab}^{-ABCD}(\boldsymbol{k}, \boldsymbol{q}, \boldsymbol{u}, \boldsymbol{v}). \tag{5.28}$$

In the new variables, the Hamiltonian simplifies to

$$H = H^{\mathrm{free}}[a] + \varepsilon(H^{\mathrm{nc}}[a] + H^{\mathrm{int}}[a]) = H^{\mathrm{free}}[\underline{a}] + \varepsilon H^{\mathrm{int}}[\underline{a}]) + \mathrm{O}(\varepsilon^2), \tag{5.29}$$

and the momentum reduces to free-particle form

$$\boldsymbol{P} = \boldsymbol{P}^{\mathrm{free}}[a] + \varepsilon \boldsymbol{P}^{\mathrm{nc}}[a] = \boldsymbol{P}^{\mathrm{free}}[\underline{a}] + \mathrm{O}(\varepsilon^2). \tag{5.30}$$

The space-like components of the angular momentum M^{ij} $(i, j = 1, 2, 3)$ also take on free-particle form, while the time-like components M^{0j} contain interaction terms. Such a structure of the canonical realization of the Poincaré group is characteristic of the instant form of relativistic dynamics [19], in which the evolution parameter t coincides with the coordinate time x^0 (as it does in our case).

6. Canonical quantization and two-particle variational states

Returning to the double index $a\alpha$ (instead of the temporarily used single one a; cf below equation (5.14)) and introducing the new complex fields variables:

$$b_a = \frac{\underline{a}_{a1}^- + \mathrm{i}\underline{a}_{a2}^-}{\sqrt{2}}, \qquad b_a^* = \frac{\underline{a}_{a1}^+ - \mathrm{i}\underline{a}_{a2}^+}{\sqrt{2}}, \qquad d_a = \frac{\underline{a}_{a1}^- - \mathrm{i}\underline{a}_{a2}^-}{\sqrt{2}}, \qquad d_a^* = \frac{\underline{a}_{a1}^+ + \mathrm{i}\underline{a}_{a2}^+}{\sqrt{2}} \tag{6.1}$$

we rewrite the Liouville form, (5.28), omitting some terms which are total differentials and thus do not contribute to the symplectic form,

$$\Theta = \mathrm{i} \sum_a^2 \int \mathrm{d}^3k \{b_a^*(\boldsymbol{k}) \tilde{\delta} b_a(\boldsymbol{k}) + d_a^*(\boldsymbol{k}) \tilde{\delta} d_a(\boldsymbol{k})\}. \tag{6.2}$$

It follows from (6.2) that $(b_a(\boldsymbol{k}), b_a^*(\boldsymbol{k}))$ and $(d_a(\boldsymbol{k}), d_a^*(\boldsymbol{k}))$ are canonically conjugate pairs. In the quantum description we consider these variables as the annihilation and creation operators $b_a(\boldsymbol{k}), b_a^\dagger(\boldsymbol{k}), d_a(\boldsymbol{k}), d_a^\dagger(\boldsymbol{k})$ satisfying the standard commutational relations and the conventional vacuum state conditions [4, 5].

The quantum operators corresponding to the Hamiltonian (5.29) and the momentum (5.30) become (with $\varepsilon = 1$):

$$H = H^{\mathrm{free}} + H^{\mathrm{int}}, \tag{6.3}$$

$$H^{\mathrm{free}} = \sum_{a=1}^2 \int \mathrm{d}^3k\, k_{a0} \left\{ b_a^\dagger(\boldsymbol{k}) b_a(\boldsymbol{k}) + d_a^\dagger(\boldsymbol{k}) d_a(\boldsymbol{k}) \right\}, \tag{6.4}$$

$$H^{\rm int} = -\sum_{a=1}^{2}\sum_{b=1}^{2}\frac{g_a g_b}{8(2\pi)^3}\int \mathrm{d}^3k\,\mathrm{d}^3q\,\mathrm{d}^3u\,\mathrm{d}^3v\,\frac{\delta^{(3)}(\boldsymbol{k}-\boldsymbol{q}+\boldsymbol{u}-\boldsymbol{v})}{\sqrt{k_{a0}q_{a0}u_{b0}v_{b0}}}$$
$$\times :\big\{d_a(\boldsymbol{k})b_a(-\boldsymbol{q}) + b_a^\dagger(-\boldsymbol{k})b_a(-\boldsymbol{q}) + d_a(\boldsymbol{k})d_a^\dagger(\boldsymbol{q}) + b_a^\dagger(-\boldsymbol{k})d_a^\dagger(\boldsymbol{q})\big\}$$
$$\times \big[\tilde K\big(u_b^+ - v_b^-\big)\big\{d_b(\boldsymbol{u})b_b(-\boldsymbol{v}) + b_b^\dagger(-\boldsymbol{u})d_b^\dagger(\boldsymbol{v})\big\}$$
$$+ \tilde K\big(u_b^+ - v_b^+\big)\big\{b_b^\dagger(-\boldsymbol{u})b_b(-\boldsymbol{v}) + d_b(\boldsymbol{u})d_b^\dagger(\boldsymbol{v})\big\}\big]:, \tag{6.5}$$

$$\boldsymbol{P} = \boldsymbol{P}^{\rm free} = \sum_{a=1}^{2}\int \mathrm{d}^3k\,\boldsymbol{k}\big\{b_a^\dagger(\boldsymbol{k})b_a(\boldsymbol{k}) + d_a^\dagger(\boldsymbol{k})d_a(\boldsymbol{k})\big\}, \tag{6.6}$$

where : : denotes normal ordering, and $u_a^\pm = (\pm u_{a0}, \boldsymbol{u})$, $v_b^\pm = (\pm v_{b0}, \boldsymbol{v})$ are 4-momenta on the upper (+) or lower (–) sheet of the mass shell.

Now we are going to consider the two-body problem. We begin with the 'particle 1 + particle 2' ansatz $|1+2\rangle = \int \mathrm{d}^3s\,\mathrm{d}^3t\,F(\boldsymbol{s},\boldsymbol{t})b_1^\dagger(\boldsymbol{s})b_2^\dagger(\boldsymbol{t})|0\rangle$, where $F(\boldsymbol{s},\boldsymbol{t})$ is a channel wavefunction. The two-particle sector of the Fock space is not invariant under the action of the Hamiltonian (6.3)–(6.5). Thus the state $|1+2\rangle$ cannot be an exact eigenstate of H whatever the function $F(\boldsymbol{s},\boldsymbol{t})$. We shall treat this function as an adjustable QFT counterpart of a variational parameter.

In the rest frame $\boldsymbol{P}|1+2\rangle = 0$ and the wavefunction takes on the form $F(\boldsymbol{s},\boldsymbol{t}) = f(\boldsymbol{s})\delta^{(3)}(\boldsymbol{s}+\boldsymbol{t})$, which follows from the free-particle structure of the momentum operator (6.6). Then using the variational principle $\delta\langle 1+2|H-E|1+2\rangle = 0$ and following [4] leads to the following equation for the wavefunction $f(\boldsymbol{s})$:

$$\left[\sum_{a=1}^{2}p_{a0} - E\right]f(\boldsymbol{p}) = \frac{g_1g_2}{8(2\pi)^3}\int\frac{\mathrm{d}^3q\,f(\boldsymbol{q})}{\sqrt{p_{10}p_{20}q_{10}q_{20}}}\sum_{a=1}^{2}\tilde K\big(p_a^+ - q_a^+\big), \tag{6.7}$$

where $\tilde K$ is defined in equation (5.24). This is a relativistic Salpeter-like wave equation for the stationary states of the two-particle system.

Similarly, using the ansatz $|1+\overline{1}\rangle = \int \mathrm{d}^3s\,\mathrm{d}^3t\,f(\boldsymbol{s})\delta^{(3)}(\boldsymbol{s}+\boldsymbol{t})b^\dagger(\boldsymbol{s})d^\dagger(\boldsymbol{t})|0\rangle$ and following [5] we arrive at the wave equation for a particle–antiparticle system (say, for particle 1 and antiparticle $\overline{1}$; we omit the subscript a since no quantities corresponding to particle 2 appear),

$$[2p_0 - E]f(\boldsymbol{p}) = \frac{g^2}{8(2\pi)^3}\int\frac{\mathrm{d}^3q\,f(\boldsymbol{q})}{p_0q_0}\{2\tilde K(p^+-q^+) + \tilde K(p^+-p^-) + \tilde K(q^+-q^-)\}, \tag{6.8}$$

where $p_0 = p_{10} = p_{20}$ and $p^\pm = p_1^\pm = p_2^\pm$ since $m = m_1 = m_2$. Note that the term $2\tilde K(p^+-q^+)$ corresponds to the exchange of one quantum of the mediating field while $\tilde K(p^+-p^-)$ and $\tilde K(q^+-q^-)$ correspond to virtual annihilation.

We now consider the nonrelativistic (NR) limit $|\boldsymbol{p}| \ll m$ of equations (6.7) and (6.8). Taking into account the fact that the symmetric Poincaré-invariant kernel $K(x)$ is a function of x^2, and thus the Fourier transform $\tilde K(p)$ is a function of p^2, we have

$$\tilde K(p^+-q^+) = \tilde K(p_0-q_0, \boldsymbol{p}-\boldsymbol{q}) \longrightarrow \tilde K(0, \boldsymbol{p}-\boldsymbol{q}), \tag{6.9}$$

$$\tilde K(p^+-p^-) = \tilde K(2p_0, \boldsymbol{0}) \longrightarrow \tilde K(2m, \boldsymbol{0}). \tag{6.10}$$

Then equation (6.7) reduces to the form

$$\left[\frac{\boldsymbol{p}^2}{2m_r} - \epsilon\right]f(\boldsymbol{p}) = \frac{g_1g_2}{4(2\pi)^3m_1m_2}\int \mathrm{d}^3q\,f(\boldsymbol{q})\tilde K(0, \boldsymbol{p}-\boldsymbol{q}), \tag{6.11}$$

where $m_r = m_1m_2/(m_1+m_2)$ and $\epsilon = E - (m_1+m_2)$.

The NR limit of the particle–antiparticle equation (6.8) similarly becomes

$$\left[\frac{\boldsymbol{p}^2}{m} - \epsilon\right] f(\boldsymbol{p}) = \frac{g^2}{4(2\pi)^3 m^2} \int \mathrm{d}^3 q\, f(\boldsymbol{q}) \{\tilde{K}(0, \boldsymbol{p} - \boldsymbol{q}) + \tilde{K}(2m, \boldsymbol{0})\}. \quad (6.12)$$

In coordinate representation these equations are the usual Schrödinger ones:

$$-\frac{1}{2m_r}\Delta\psi(\boldsymbol{r}) + \frac{g_1 g_2}{4m_1 m_2} U(r)\psi(\boldsymbol{r}) = \epsilon\psi(\boldsymbol{r}) \quad (6.13)$$

for (6.11), and

$$-\frac{1}{m}\Delta\psi(\boldsymbol{r}) + \frac{g^2}{4m^2}\{U(r) - \tilde{K}(2m, \boldsymbol{0})\delta^{(3)}(\boldsymbol{r})\}\psi(\boldsymbol{r}) = \epsilon\psi(\boldsymbol{r}) \quad (6.14)$$

for (6.12). The function $U(r)$ can be obtained from either the momentum or coordinate representations of the kernels:

$$U(r) = -\int \frac{\mathrm{d}^3 k}{(2\pi)^3}\, \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{x}} \tilde{K}(0, \boldsymbol{k}) = -\int \mathrm{d}x^0 K(x). \quad (6.15)$$

For the Yukawa model the kernel in the momentum representation has the form:

$$\tilde{K}(k) = \tilde{D}_0(k; \mu) = \int \mathrm{d}^4 x\, \mathrm{e}^{-\mathrm{i}k\cdot x} D_0(x; \mu) = \mathcal{P}\frac{1}{\mu^2 - k^2} \quad (6.16)$$

where $D_0(x; \mu)$ is the symmetric Green function of the Klein–Gordon equation [18]. Thus, the function $U(r)$ and the constant $\tilde{K}(2m, \boldsymbol{0})$ become

$$U_{\mathrm{Y}}(r) = -\frac{1}{4\pi r}\, \mathrm{e}^{-\mu r}, \qquad \tilde{K}(2m, \boldsymbol{0}) = \frac{1}{\mu^2 - 4m^2}. \quad (6.17)$$

We note that

$$(p^+ - q^+)^2 \begin{cases} = 0 & \text{if} \quad \boldsymbol{p} = \boldsymbol{q} \\ < 0 & \text{otherwise,} \end{cases} \qquad (p^+ - p^-)^2 \geqslant 4m^2. \quad (6.18)$$

Thus the on-shell value $\tilde{K}(p^+ - q^+)$ of the kernel (6.16) is a non-singular function, and the principal value sign $\mathcal{P}$ for such terms of equations (6.7) and (6.8) can be omitted. The same applies to the annihilation terms $\tilde{K}(p^+ - p^-)$ and $\tilde{K}(q^+ - q^-)$ unless $\mu \geqslant 2m$, in which case the principal value should be taken to avoid the singularity at $|\boldsymbol{p}| = \sqrt{\mu^2 - 4m^2}$.

Let us consider the case of the tachyonic mediating field which corresponds to the Klein–Gordon field with imaginary mass $\mu = \mathrm{i}\varkappa$. The quantized theory of tachyons encounters difficulties in the description of free quanta [20–22]; however, systems with free quanta are not considered here. In contrast to the standard Yukawa model, the on-shell tachyonic kernel $\tilde{K}(p^+ - q^+)$ possesses a singularity and the principal value should be taken on the rhs of equations (6.7) and (6.8). The annihilation terms of equation (6.8) (containing $\tilde{K}(p^+ - p^-)$ and $\tilde{K}(q^+ - q^-)$) are regular everywhere in this case. The tachyonic NR potential is a long-range one since it differs from the Yukawa one (6.17) by the oscillating numerator $\cos \varkappa r$, instead of $\exp(-\mu r)$.

In the next section we consider some kernels arising from nonstandard field theories.

7. Kernels arising from nonstandard field theories

Various nonstandard field theories, which may lead to difficulties in quantized form, are nevertheless used as effective field theories. Examples are field theories with higher derivatives [22–24] and nonlocal field theories [10–12]. Among the problems are those connected to the consistent inclusion of free field quanta in the theory. On the other hand, if a nonstandard

field is used only as a mediator of the interaction (and free quanta are absent), the theory may correspond to a system of physical interest. The most radical realization of this idea is the virton theory [12] where nonlocal field equations possess no solution for free quanta but the Green function and thus the mediated interaction is not trivial. The quantization of the virton field is a rather elaborate procedure. In contrast, the present scheme, where mediating fields are eliminated at the classical level, looks to be the simplest way to extract useful content from nonstandard field theories. The output of the scheme is the covariant Green function for the corresponding field equation. We shall consider the general structure and a few examples of kernels for nonstandard field theories.

7.1. General structure of kernels

Let the Lagrangian density (henceforth 'Lagrangian') of the free χ-field be $\mathcal{L}_\chi = \frac{1}{2}\chi F(\Box)\chi$. Then the corresponding field equation (instead of (2.3)) reads

$$F(\Box)\chi = \rho. \tag{7.1}$$

The 'elementary solution' of this equation, i.e., by definition [25], the solution of the equation

$$F(\Box)K(x) = \delta^{(4)}(x), \tag{7.2}$$

takes on the formal expression:

$$K(x) = F^{-1}(\Box)\delta^{(4)}(x) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\,\mathrm{e}^{\mathrm{i}k\cdot x}\tilde{K}(k), \qquad \text{where} \quad \tilde{K}(k) = 1/F(-k^2). \tag{7.3}$$

The existence and properties of $K(x)$ depend of the choice of the operator $F(\Box)$. Note that in general $K(x)$ is not a Green function in the conventional sense since Cauchy's problem is not well posed [10, 20]. For this reason we shall use the term 'elementary solution' [25] rather than 'Green function'.

Of physical interest are theories where $F(z)$ is the integer function of the complex variables z, such that $[F(z)]^* = F(z^*)$ (the latter equality implies that $\mathcal{L}_\chi$ is real). Then the operator $F(\Box)$ can be presented in the form [10]

$$F(\Box) = \text{const}\,\mathrm{e}^{f(\Box)}\prod_{n=1}^{N}\left(\Box + \mu_n^2\right), \tag{7.4}$$

where $f(z)$ is an integer function (so that $\mathrm{e}^{f(z)}$ has no zeros), parameters μ_n^2 are real or complex (whereupon there are also factors with $[\mu_n^2]^*$ in the product), and N may be infinite.

If the operator $F(\Box)$ is purely polynomial (i.e., $f(\Box) = 0$) the field equation (7.1) is a higher derivative differential equation. Then, in general, the χ-field can be split effectively into N Klein–Gordon fields of masses μ_n which, although they contribute with different signs into the total energy, cancel mutually field singularities of point-like sources [10]. Of special interest are degenerate cases which may exhibit fundamentally different features from the Klein–Gordon case. In the next subsection we will consider the choice $F(\Box) = \Box^2$ which leads to a confining interaction.

If $f(\Box) \neq 0$ the field equation (7.1) is an integral one. Usually the operator $\mathrm{e}^{f(\Box)}$ is used in nonlocal field theories [10, 12] and also in related regularization methods [13, 15, 16] in order to suppress ultraviolet divergences. For this purpose the function $f(z)$ which is increasing as $z \to +\infty$ is appropriate. For example, the following choice of nonlocal field operator is used often in the literature [10, 15, 16]:

$$F(\Box) = \mathrm{e}^{(\Box+\mu^2)/\Lambda^2}(\Box + \mu^2), \tag{7.5}$$

where Λ is a cut-off parameter. The corresponding symmetrical kernel,

$$\tilde{K}(k) = \mathcal{P}\frac{\mathrm{e}^{(k^2-\mu^2)/\Lambda^2}}{\mu^2 - k^2}, \tag{7.6}$$

decreases exponentially as $k^2 \to -\infty$. This leads to the NR potential,

$$U(r) = -\frac{1}{8\pi r}\sum_{\pm} \mathrm{e}^{\pm\mu r}[\mathrm{erf}(\Lambda r/2 \pm \mu/\Lambda) \mp 1], \tag{7.7}$$

which coincides with the Yukawa potential (6.17) for $r \gg 1/\Lambda$ and is finite at $r = 0$. The on-shell value $\tilde{K}(p^+ - q^+)$ of the kernel (7.6) is regular and exponentially decreasing at $(\boldsymbol{p} - \boldsymbol{q})^2 \to \infty$, so that the two-particle equation (6.7) is solvable.

In contrast to the two-particle (i.e., particle 1 + particle 2) case, the particle–antiparticle problem is determined by the behaviour of kernel $\tilde{K}(k)$ not only in the domain $k^2 < 0$, but also in $k^2 > 0$. In turn, this depends on the properties of the cut-off function $\mathrm{e}^{f(z)}$ at $z < 0$. Following [12], the cut-off function must possess a power asymptotics at $z \to -\infty$. This is not the case for the choice (7.5), (7.6) which leads to difficulties in the particle–antiparticle problem. Indeed, the function $\tilde{K}(p^+ - p^-)$ is exponentially increasing at $\boldsymbol{p}^2 \to \infty$ so that equation (6.8) is inconsistent.

Finally, if the operator (7.4) has no polynomial factor, i.e., $F(\Box) = \mathrm{e}^{f(\Box)}$, the homogeneous equation $F(\Box)\chi_0 = 0$ possesses a trivial solution $\chi_0 = 0$ only . This case is typical of virton field theory [12]. The previous remark concerning the choice of cut-off function is actual in this case also.

7.2. Higher derivative theory for quark binding

Quantum chromodynamics is based on classical Yang–Mills equations, but no solutions of these equations, exhibiting confinement, have appeared in the literature to date. Moreover, such classical confining solutions are believed not to exist, since confinement is felt to be a quantum phenomenon. The quantum (QCD) description of confinement has proved to be very challenging and has still not been entirely resolved.

On the other hand, a number of comparatively simple theories with higher derivatives are known, which, despite quantization difficulties, lead to confinement—even at the classical level [23, 24]. Theories of similar but non-Abelian structure have arisen from QCD as effective theories of the gluon field in infrared asymptotics [26–28]. Classical Abelian solutions [29] to effective field equations [27] have been shown to provide confinement in the relativistic potential model [29, 30]. Here, we combine the simplest such theory [23] with the generalized Yukawa model.

The free-χ term of the Lagrangian can be written as

$$\mathcal{L}_\chi = \frac{1}{2\varkappa^2}(\Box\chi)^2 \simeq \frac{1}{2\varkappa^2}(\partial_\mu\partial_\nu\chi)(\partial^\mu\partial^\nu\chi), \tag{7.8}$$

where $\varkappa$ is a constant with dimensions of mass, and $\simeq$ denotes equality modulo divergence terms. The equation for the mediating field is the fourth-order equation,

$$\Box^2\chi = -\varkappa^2\rho. \tag{7.9}$$

We need the symmetric elementary solution $E(x)$ of (7.9), which satisfies the equation

$$\Box^2 E(x) = \delta^{(4)}(x). \tag{7.10}$$

It has the form [29]

$$E_0(x) = \frac{1}{16\pi}\theta(x^2), \tag{7.11}$$

where an arbitrary constant C can be added to the rhs. Thence the solution of (7.9),

$$\chi = K * \rho, \qquad \text{with the kernel} \quad K(x) = -\varkappa^2 E_0(x), \tag{7.12}$$

is used in the original action $I = \int \mathrm{d}^4x \mathcal{L}(x)$ for the construction of the reduced theory. The choice $C = -1/(16\pi)$ permits us to calculate the NR potential without the need for regularization which would otherwise be required. Using equation (6.15) we obtain the result

$$U_{\mathrm{L}}(r) = \frac{\varkappa^2 r}{8\pi}. \tag{7.13}$$

The higher-order Lagrangian theory can be reformulated as an equivalent one with a first-order Lagrangian but depending on two fields [10, 23]. The term (7.8) in the Lagrangian $\mathcal{L} = \sum_{a=1}^{2} \mathcal{L}_a + \mathcal{L}_\chi + \mathcal{L}_{\mathrm{Y}}$ is replaced by

$$\mathcal{L}_{\phi\chi} = (\partial_\mu \varphi)(\partial^\mu \chi) - \tfrac{1}{2}\varkappa^2 \varphi^2, \tag{7.14}$$

which leads to the pair of coupled inhomogeneous d'Alembert equations,

$$\Box\chi = -\varkappa^2 \varphi \tag{7.15}$$

$$\Box\varphi = \rho. \tag{7.16}$$

Let us construct the time-symmetric solution to these equations. For (7.16) we have

$$\varphi = D_0 * \rho, \qquad \text{with the kernel} \quad D_0(x) = \tfrac{1}{2}\{D_+(x) + D_-(x)\}, \tag{7.17}$$

where

$$D_\eta(x) = \frac{1}{4\pi}\{1 + \eta \operatorname{sign} x^0\}\delta(x^2), \qquad \eta = \pm 1, 0 \tag{7.18}$$

are the retarded (if $\eta = +$), the advanced (if $\eta = -$) and the symmetric (if $\eta = 0$) Green functions of the d'Alembert equation. Once the rhs of equation (7.15) is known, the solution of this equation may be sought in the same way. However such a solution does not exist. It has the formal expression $\chi = -\varkappa^2 D_0 * \varphi = -\varkappa^2 D_0 * D_0 * \rho = -\frac{1}{4}\varkappa^2 (D_+ + D_-) * (D_+ + D_-) * \rho$, which is divergent since $D_+ * D_-$ is not a well defined distribution [29]. Instead, the function χ, defined by equations (7.12) and (7.11), together with φ, equation (7.17), does satisfy the set of field equations (7.15) and (7.16). This can be seen from the following representation [29]:

$$E_0(x) = \tfrac{1}{2}[D_+ * D_+ + D_- * D_-](x); \tag{7.19}$$

then $\Box\chi = -\frac{1}{2}\varkappa^2 \Box(D_+ * D_+ + D_- * D_- - 1/(8\pi)) * \rho = -\varkappa^2 \frac{1}{2}(D_+ + D_-) * \rho = -\varkappa^2 \varphi$.

The elimination of the fields φ and χ from the Lagrangian $\mathcal{L} = \sum_{a=1}^{2} \mathcal{L}_a + \mathcal{L}_\chi + \mathcal{L}_{\mathrm{Y}}$, where $\mathcal{L}_\chi$ is defined by equation (7.14), yields the reduced Lagrangian.

Let us consider the Salpeter-like equations (6.7) and (6.8) for this case. The construction of the Fourier transform $\tilde{K}(k)$ of the kernel $K(x) = -\varkappa^2 E_0(x)$ is a subtle problem. From naive dimensional arguments we have $\tilde{K}(k) \sim -\varkappa^2/k^4$, i.e., there is a strong singularity on the light cone $k^2 = 0$. Thus this kernel should be properly defined.

One can proceed from the analytical continuation method. Following [25] the generalized functions $(k^2 \pm \mathrm{i}0)^\lambda$ as distributions in 4D k-space and analytical functions of λ have a pole at $\lambda = -2$. Then using the Fourier transformation and the Laurent series expansion one can present the Fourier transform of $E_0(x)$ as follows:

$$\tilde{E}_0(k) = \lim_{\varepsilon \to 0} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\varepsilon[(k^2 - \mathrm{i}0)^{-2+\varepsilon} + (k^2 + \mathrm{i}0)^{-2+\varepsilon}]. \tag{7.20}$$

A similar method (in combination with the Wick rotation) was used in [31] for calculation of the nonrelativistic limit of the Bethe–Salpeter equation with confining kernel.

In our case equations (6.7) and (6.8) include the on-shell value $\tilde{K}(p^+ - q^+)$ of the kernel which is a function in 3D momentum space (not in 4D, as in the Bethe–Salpeter case). The confining kernel is highly singular at $\boldsymbol{p} = \boldsymbol{q}$, as follows from (6.18). The existence of the on-shell value $\tilde{E}_0(p^+ - q^+)$ of the function (7.20) as a distribution in 3D space has not been studied as yet.

Thus we refer to another method of regularization of confining potentials in the momentum representation which was elaborated in a number of works [32–34]. The idea of the method originates from nonrelativistic considerations [33] and is based on the observation that $U_{\mathrm{L}} = -\frac{1}{2}\varkappa^2 \lim_{\mu\to 0} \partial^2 U_{\mathrm{Y}}/\partial\mu^2$. This relation applied to the Fourier transform of potentials defines the regularized confining potential in momentum space [33].

It appears that a similar relation exists for corresponding covariant functions,

$$E_0(x) = -\tfrac{1}{2} \lim_{\mu\to 0} \partial^2 D_0(x;\mu)/\partial\mu^2, \tag{7.21}$$

where $E_0(x)$ is defined by equation (7.11) and $D_0(x;\mu)$ is the symmetric Green function of the Klein–Gordon equation. The same relation in the momentum representation,

$$\tilde{E}_0(k) = -\frac{1}{2} \lim_{\mu\to 0} \frac{\partial^2}{\partial\mu^2} \tilde{D}_0(k;\mu) = \lim_{\mu\to 0} \left\{ \frac{1}{(\mu^2 - k^2)^2} - \frac{4\mu^2}{(\mu^2 - k^2)^3} \right\}, \tag{7.22}$$

can be considered as the definition of the Fourier transform $\tilde{E}_0(k)$ of the function $E_0(x)$. Moreover, the on-shell value $\tilde{E}_0(p^+ - q^+)$ of the function (7.22) is a well-defined distribution in 3D space. We demonstrate this below, following [34] where relativistic generalizations of confining potentials in momentum space are analysed.

Decomposing the wavefunction $f(\boldsymbol{p})$ into partial waves $f_\ell(\mathrm{p})$, where $\mathrm{p} = |\boldsymbol{p}|$, one can present singular interaction terms of equations (6.7) and (6.8) as $\lim_{\mu\to 0} I_\mu = \lim_{\mu\to 0} (\mathrm{const}) A(\mathrm{p}) \frac{\partial}{\partial\mu}[\mu\Delta_\mu]$, where $\Delta_\mu = \int_0^\infty \mathrm{dq}\, f_\ell(\mathrm{q}) \mathrm{B}(\mathrm{q}) \frac{Q'_\ell(z_\mu)}{\mathrm{pq}}$, the functions $A(\mathrm{p}) = \mathrm{B}(\mathrm{p}) = (\mathrm{p}_{10}\mathrm{p}_{20})^{-1/2}$ are smooth everywhere, $z_\mu = \left(p_{a0}q_{a0} - m_a^2 + \frac{1}{2}\mu^2\right)\big/\mathrm{pq}$, and $Q_\ell(z)$ is the Legengre function of second kind. Hence, as discussed in section 3 and appendix A of [34], $\lim_{\mu\to 0} I_\mu$ exists.

The annihilation terms have no singularities (cf (6.18)).

7.3. Accounting for short-range interactions

In order to obtain, in the NR limit, the funnel potential rather than the linear one (7.13) we modify the previous model, namely the free-χ Lagrangian (7.8), as follows:

$$\mathcal{L}_\chi \longrightarrow \bar{\mathcal{L}}_\chi = \frac{1}{2\varkappa^2}(\Box\chi - \rho)^2. \tag{7.23}$$

Now the interaction comes not only from the Yukawa term $\mathcal{L}_{\mathrm{Y}}$ but also from $\bar{\mathcal{L}}_\chi$. The field equations take on the form

$$\left(\Box + m_a^2\right)\phi_a = g_a\phi_a \left\{ \frac{1}{2\varkappa^2} \left(\Box\chi - \frac{1}{2}\rho \right) - \chi \right\}, \tag{7.24}$$

$$\Box^2\chi = -\left(\varkappa^2 - \frac{1}{2}\Box \right)\rho \tag{7.25}$$

(the equation for ϕ_a^* is evidently similar to (7.24)). Omitting details, we write down the symmetric solution of equation (7.25):

$$\chi = -\left(\varkappa^2 E_0 - \tfrac{1}{2} D_0\right) * \rho. \tag{7.26}$$

The use of this solution on the rhs of equations (7.24) reduces them to the following:

$$\left(\Box + m_a^2\right)\phi_a = g_a\phi_a(\varkappa^2 E_0 - D_0) * \rho. \tag{7.27}$$

One can construct the reduced Lagrangian as discussed earlier. It has the form (2.5), but with the kernel

$$K = -\varkappa^2 E_0 + D_0, \tag{7.28}$$

which generates the NR funnel potential:

$$U_{\mathrm{F}}(r) = U_{\mathrm{L}}(r) + U_{\mathrm{C}}(r) = \frac{\varkappa^2 r}{8\pi} - \frac{1}{4\pi r}. \tag{7.29}$$

The reduced Lagrangian obviously reproduces equations (7.27).

Similar to the previous example, we can consider an equivalent formulation of the present model in terms of the first-order Lagrangian:

$$\mathcal{L} = \sum_{a=1}^{2} \mathcal{L}_a + (\partial_\mu \varphi)(\partial^\mu \chi) - \frac{1}{2}\varkappa^2 \varphi^2 + \rho\left(\chi + \frac{1}{2}\varphi\right). \tag{7.30}$$

After the change of variables $\bar{\chi} = \chi + \frac{1}{2}\varphi$, $\bar{\varphi} = \chi - \frac{1}{2}\varphi$, this Lagrangian takes on the more conventional form

$$\mathcal{L} = \sum_{a=1}^{2} \mathcal{L}_a + \frac{1}{2}(\partial_\mu \bar{\chi})(\partial^\mu \bar{\chi}) - \frac{1}{2}(\partial_\mu \bar{\varphi})(\partial^\mu \bar{\varphi}) - \frac{1}{2}\varkappa^2(\bar{\chi} - \bar{\varphi})^2 + \rho\bar{\chi}, \tag{7.31}$$

where the matter fields ϕ_a interact only with the mediating field $\bar{\chi}$ via the Yukawa term. Note that if we put $\bar{\phi} = 0$ in the Lagrangian (7.31), the model reduces to the conventional Yukawa model, with a Yukawa potential (6.17) in the NR limit. Otherwise, if $\bar{\chi} = 0$, the interaction disappears and ϕ contributes in the Lagrangian (7.31) as a free tachyonic field. But neither $\bar{\phi} = 0$, $\bar{\chi} \neq 0$ nor $\bar{\phi} \neq 0$, $\bar{\chi} = 0$ are solutions of the variational problem for the Lagrangian (7.31). It is also evident that the energy of the system of fields described by the Lagrangian (7.31) is not necessarily positive. This fact is the source of difficulties when the quantum theory with free mediating fields is constructed [10].

8. Conclusions

Although the partially reduced formulation of QFT was introduced several years ago, it has been realized recently that this approach is a time nonlocal field theory. The main idea of the present paper is the treatment of time nonlocality by means of an appropriate Hamiltonian formalism [8]. The Hamiltonization procedure is dependent in an essential way on the explicit time evolution of the fields. Since the Yukawa-like systems are not solvable exactly, we are forced to use an appropriate approximation scheme, such as the coupling constant expansions of the present paper. It turns out that two-body equations obtained in previous works [4, 5], that use the reduced formulation of the Yukawa model, correspond to the first-order approximation in the coupling constant expansion (see section 8).

In order to push the approach to higher-order approximations it is important, above all, to have a proper procedure of reducing the Liouville form to canonical form rather than some heuristic prescription. Another possibility is the development of some noncanonical quantization procedure. Either way, higher-order approximations need to be compared to conventional perturbative QFT.

Nevertheless, even within the first-order approximation, the partially reduced approach leads simply to Salpeter-like two-body wave equations, which may describe a wide class of interactions. The kernels of such equations can be related to standard field theories, or to various nonstandard classical field theories which describe characteristic features of the interaction under consideration in an effective way.

The approach can be applied to systems of fields with nonzero spin, such as fermionic matter fields and vector or tensor mediating fields.

Acknowledgment

The support of the NATO Collaborative Linkage Grant programme for this work is gratefully acknowledged.

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